6

Classical Mechanics

6.1. Mechanics of a system of Particles

(a) Conservation theorem for linear momentum :

The net linear momentum of a system of n-particles is

$$\mathbf{P} = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m_i v_i$$

From Newton's second law, $\mathbf{F}^{ext} = \frac{d\mathbf{P}}{dt}$

i.e., the rate of change of linear momentum of a system of particles is equal to the net external force acting on the system.

If $\quad \mathbf{F}^{ext} = 0, \frac{d\mathbf{P}}{dt} = 0.$ Integrating, $\mathbf{P} =$ constant.

This gives the *theorem* for conservation of linear momentum of the system according to which "*If the sum of external forces acting on the system of particles is zero, the total linear momentum of the system is constant or conserved.*"

(b) Conservation theorem for angular momentum.

The angular momentum of ith particle of the system about any point O, from definition is given by

$$\mathbf{J}_i = \mathbf{r}_i \times \mathbf{p}_i, \qquad \text{... (1)}$$

where $\mathbf{r}_i$ is the radius vector of ith particle from the point O and $\mathbf{p}_i$ its linear momentum.

For a *system of n particles*, we have

$$\mathbf{J} = \sum_i \mathbf{J}_i = \sum_i \mathbf{r}_i \times \mathbf{p}_i \qquad \text{...(2)}$$

$$\frac{d\mathbf{J}}{dt} = \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} = \sum_i \mathbf{r}_i \times \mathbf{F}_i \qquad \left(\because \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i = \mathbf{v}_i \times \mathbf{p}_i = 0 \right)$$

Here, $\mathbf{F}_i = \frac{d}{dt}\mathbf{p}_i =$ net force acting on i^{th} particle.

Internal forces occur in equal and opposite pairs. Hence the net internal force acting on the system of particles is zero. Thus,

$$\frac{d\mathbf{J}}{dt} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} = \tau^{ext}$$

Here, $\tau^{ext} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext}$ is the torque arising due to external forces only.

If $\tau^{ext} = 0, \frac{d\mathbf{J}}{dt} = 0$ or $\mathbf{J} =$ constant.

Thus, *if external torque acting on a system of particles is zero, the angular momentum of the system remains constant.*

This is the conservation theorem for angular momentum of a system of particles.

(c) Conservation of Energy. If the work done by a force is independent of path, the force is said to be conservative.

If the forces acting on the system of particles are conservative, the total energy of the system of particles which is the sum of the total kinetic energy and the total potential energy of the system is conserved.

This is the energy conservation theorem.

On the other hand *if the forces are non-conservative, the total energy of universe* (mechanical energy + chemical energy + sound energy + light energy + heat energy etc.) *remains constant.*

6.2. Basic Concepts

Degrees of Freedom. *The number of mutually independent variables required to define the state or position of a system is the number of degrees of freedom possessed by it.*

For example, the position of a simple ideal mass-point can be defined completely by the three cartesian coordinates x,y,z. So it has three degrees of freedom. Extending this idea, for a system of N particles moving independently of each other, the number of degrees of freedom is $3N$.

Constraints. *Constraints are restrictions imposed on the position or motion of a system, because of geometrical conditions.*

Examples. (1) The beads of an abacus are constrained to one-dimensional motion by the supporting wires.

(2) Gas molecules within a container are constrained by the walls of the vessel to move only *inside* the container.

(3) The motion of rigid bodies is always such that the distance between any two particles remains unchanged.

(4) A particle placed on the surface of a solid sphere is restricted by the constraint so that it can only move on the surface or in the region exterior to the sphere.

For a particle constrained to move on a plane, only two variables x, y or r, θ *are sufficient to describe its motion and the particle is said to have two degrees of freedom. Thus, the constraint on the motion of the particle in a plane reduces the number of degrees of freedom by one.*

Very often, we can express constraints in terms of certain equations. For example, the equation of constraint in the case of a particle moving on or outside the surface of a sphere of radius a is $x^2 + y^2 + z^2 \geq a^2$ if the origin of the coordinate system coincides with the centre of the sphre.

(i) Holonomic and non-holonomic constraints.

If the constraints can be expressed as equations connecting the co-ordinates of the particles (and possibly time) in the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_n, t) = 0 \qquad \ldots (1)$$

then the constraints are said to be *holonomic*.

Examples. (1) *The constraints involved in the motion of rigid bodies* in which the distance between any two particular points is always fixed, are *holonomic* since the conditions of constrains are expressed as

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0.$$

(2) *The constraints involved* when a particle is restricted to move along a curve or surface are holonomic. Here the equation defining the curve or surface is the equation of constraint.

If the constraints cannot be expressed in the form of Eq. (1), they are called *non-holonomic constraints*.

Examples. (1) The constraints involved in the motion of the particle placed on the surface of a solid sphere are non- holonomic. The conditions of constraints in this case are expressed as

$$r^2 - a^2 \geq 0,$$

where a is the radius of sphere. This is an inequality and hence not in the form of Eq. (1).

(2) The walls of the gas vessel constitute a nonholonomic constraint.

(3) An object rolling on a rough surface without slipping is also an example of non-holonomic constraint.

(ii) **Scleronomic and Rheonomic Constraints.** If the constraints are independent of time, they are called *scleronomic*.

If the constraints are explicitly dependent on time, they are called *rheonomic*.

The constraint in the case of rigid body motion is scleronomous. A bead sliding on a moving wire is an example of rheonomic constraint.

In the solution of mechanical problems, the constraints introduce two types of difficulties :

(1) The co-ordinates $\mathbf{r}_i$ are connected by the equations of constraints. Therefore, they are not independent.

(2) The forces of constraint are not *a priori* known. In fact, they cannot be estimated till a complete solution of the problem is obtained.

The first problem can be solved by introducing *generalized coordinates*, whereas the second is practically an insurmountable problem. We therefore reformulate the problem such that the forces of constraint disappear.

6.3 Generalised co-ordinates.

A system consisting of N particles, free from constraints, has $3N$ independent coordinates or degrees of freedom. If the sum of the degrees of freedom of all the particles is k, then the system may be regarded as a collection of free particles subjected to $(3N-k)$ independent constraints. So only k coordinates are needed to describe the motion of the system. These new coordinates $q_1, q_2, q_3 \ldots q_k$ are called the *Generalised Coordinates of Lagrange*. Generalised coordinates may be lengths or angles or any other set of independent quantities which define the position of the system.

Definition. *The generalised coordinates of a material system are the independent parameters $q_1, q_2, q_3, \ldots q_k$ which completely specify the configuration of the system, i.e., the position of all its particles with respect to the frame of reference.*

Example. Consider the simple pendulum of mass m_1 with fixed length r_1 (Fig. 6.1). The single coordinate θ_1 will determine uniquely the position of m_1 since the simple pendulum is a system of one degree of freedom. Since the only variable involved is θ_1, it can be chosen as the generalised coordinate. Thus $q = \theta_1$. The two coordinates x_1 and y_1 could also be used to locate m_1 but would require the inclusion of the *equation of the constraint* $x_1^2 + y_1^2 = r_1^2$. Since x_1 and y_1 are not independent, they are not generalised coordinates.

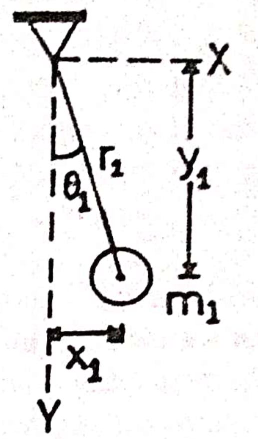

Fig. 6.1

Generalised velocities : The generalised velocities of a system are the total time derivatives of the generalised coordinates of the system.

Thus $$\dot{q}_i = \frac{dq_i}{dt} \quad (i = 1, 2, 3, \ldots, k).$$

6.4 Transformation equations.

The rectangular cartesian co-ordinates can be expressed as the functions of generalised co-ordinates. Let x_i, y_i and z_i be the cartesian coordinates of ith particle of the system. Let t denote the time. Then, these cartesian co-ordinates can be expressed as functions of generalised co-ordinates $q_1, q_2, q_3, \ldots, q_k$ *i.e.*,

$$\left.\begin{aligned} x_i &= x_i(q_1, q_2, \ldots, q_k, t) \\ y_i &= y_i(q_1, q_2, \ldots, q_k, t) \\ z_i &= z_i(q_1, q_2, \ldots, q_k, t) \end{aligned}\right\} \quad \ldots(1)$$

Let $\mathbf{r}_i$ be the position vector of ith particle, *i.e.*, $\mathbf{r}_i = \mathbf{i}x_i + \mathbf{j}y_i + \mathbf{k}z_i$.

Then $$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, q_3, \ldots, q_k, t), \quad \ldots(2)$$

Eq. (2) is the vector form of Eq. (1).

The equations like (1) and (2) are called *transformation equations*. The functions and their derivatives in the above two equations are supposed to be continuous. The equations also contain the constraints explicitly.

6.5. Configuration space

In the case of motion of a single particle we can represent its trajectory in the three dimensional space by specifying its variables. For a system of N particles described by $3N$ space coordinates with $(3N - k)$ equations of constraint in the real space, it is difficult to visualise the motion of the entire system. It is convenient to describe the motion of a system having k coordinates in a hypothetical k dimensional space. The instantaneous configuration of the system is described by the values of the k generalised coordinates $q_1, q_2, q_3, \ldots, q_k$, and corresponds to a particular point in a cartesian hyperspace where the q's form the k coordinate axes. The point is called the system point and the k dimensional space is known as the *Configuration space.* At some later instant, the state of the system changes and it will be represented by some other point in the configuration space. Thus, the system point moves in the configuration space tracing out a curve. This curve represents "the path of motion of the system". "The motion of the system", as used above, then refers to the motion of the system point along this path in *configuration space.* Time can be considered formally as a parameter of the curve since each point in the configuration space has one or more values of time associated with it. *Configuration space has nothing in common with the three-dimensional space which we can visualise physically. It is a purely geometric structure by means of which the laws of the variation of the state of a system can be formulated in geometrical language.*

6. 6. Principle of virtual work

Consider a system described by n generalized coordinates q_j $(j = 1, 2, 3, \ldots, n)$. Suppose the system undergoes a certain displacement in the configuration space in such a way that it does not take any time and that it is consistent with the constraints on the system. Such displacements are called *virtual* because they do not represent actual displacements of the system. Since there is no actual motion of the system, the work done by the forces of constraint in such a virtual displacement is zero.

Let the virtual displacement of the ith particle of the given system be $\delta\mathbf{r}_i$. If the given system is in equilibrium, the resultant force acting on the ith particle of the system must be zero, *i.e.*, $\mathbf{F}_i = 0$. It is, then, obvious that *virtual work* $\mathbf{F}_i \cdot \delta\mathbf{r}_i = 0$ for the ith particle and hence it is also zero for all the particles of the system.

Thus
$$dW = \sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0 \qquad \ldots (1)$$

The resultant force $\mathbf{F}_i$ acting on the ith particle is

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i \qquad \ldots (2)$$

Here, $\mathbf{F}_i^a$ is the applied force and $\mathbf{f}_i$ is the force of constraint.

Eq. (1) then becomes

$$\sum_i \mathbf{F}_i^a \cdot \delta\mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0 \qquad \ldots (3)$$

We now consider systems for which the virtual work done by the forces of constraints is zero, *i.e.*,

$$\sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0 \qquad ...(4)$$

Then Eq. (3) becomes

$$\sum_i \mathbf{F}_i^a \cdot \delta\mathbf{r}_i = 0 \qquad ...(5)$$

This equation is termed as *principle of virtual work.*

6.7. D' Alembert's Principle

Most of the systems we come across in mechanics are not in static equilibrium. Hence the principle must be modified to include dynamic systems as well. According to Newton's second law of motion,

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \quad \text{or} \quad \mathbf{F}_i - \dot{\mathbf{p}}_i = 0 \qquad ...(6)$$

According to the above equation, a moving system of particles can be considered to be in equilibrium under the force $(\mathbf{F}_i - \dot{\mathbf{p}}_i)$,*i.e.*, the actual applied force $\mathbf{F}_i$ plus an additional force $-\dot{\mathbf{p}}_i$ which is known as *reversed effective force* on ith particle. Let us again assume that the *forces of constraint do no work.* Then, we can generalize Eq. (5) by the use of Eq. (6) to the form

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta\,\mathbf{r}_i = 0 \qquad ...(7)$$

Eq. (7) is the mathematical statement of D'Alembert's principle.

It is to be noted here that we have restricted ourselves to the systems where the virtual work done by the forces of constraints disappears. With this in mind we can drop the superscript a in equation (2) *i.e.*, D'Alembert's principle may be written as

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0. \qquad ...(8)$$

6.8. Derivation of Lagrange's Equation of Motion

Lagrange's Equations from D'Alembert's Principle

Consider a system of particles whose position vectors are expressed as functions of generalized coordinates $q_1, q_2, q_3, ...q_k ...q_f$ and the time t.

Consider any particle of the system (ith particle) of mass m_i and acted upon by an external force $\mathbf{F}_i$.

According to D' Alembert's principle,

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0 \qquad ...(1)$$

Here $\dot{\mathbf{p}}_i$ is the inertial force for ith particle and $\delta\mathbf{r}_i$ is the virtual displacement of ith particle due to action of force $\mathbf{F}_i$.

In general, $\quad \mathbf{r}_i = \mathbf{r}_i\,(q_1, q_2, q_3, \ldots q_k, \ldots q_f, t)$. ...(2)

$$\dot{\mathbf{r}}_i = \frac{\delta \mathbf{r}_i}{\delta t} = \frac{\partial \mathbf{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \mathbf{r}_i}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \ldots + \frac{\partial \mathbf{r}_i}{\partial q_f}\dot{q}_f + \frac{\partial \mathbf{r}_i}{\partial t}$$

$$= \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \qquad \ldots(3)$$

The virtual displacement $\delta \mathbf{r}_i$ in terms of generalised co-ordinates is given by

$$\delta \mathbf{r}_i = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k. \qquad \ldots(4)$$

Now, $\dot{\mathbf{p}}_i = m_i \ddot{\mathbf{r}}_i$

Therefore Eq. (1) becomes $\sum_i (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0$. ...(5)

or

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i. \qquad \ldots(6)$$

Putting the value of $\delta \mathbf{r}_i$ from Eq. (4) in (6), we get

$$\sum_i \sum_k m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k \qquad \ldots(7)$$

Now,

$$\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\right) = \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} + \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)$$

$$\therefore \quad \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\right) - \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right) \qquad \ldots(8)$$

Putting the value of $\ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$ from above equation in (7), we get

$$\sum_i \sum_k m_i \left[\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\right) - \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)\right]\delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)\delta q_k \qquad \ldots(9)$$

Differentiating Eq. (3) partially with respect to $\dot{q}_k$,

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} = \frac{\partial \mathbf{r}_i}{\partial q_k} \qquad \ldots(10)$$

Differentiating Eq. (3) partially with respect to q_k,

$$\frac{\partial \dot{\mathbf{r}}_1}{\partial q_k} = \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_1}\dot{q}_1 + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_2}\dot{q}_2 + \ldots + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_k}\dot{q}_k + \ldots + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial t} \qquad \ldots(11)$$

Also we have

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right) = \frac{\partial}{\partial q_1}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)\dot{q}_1 + \frac{\partial}{\partial q_2}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)\dot{q}_2 + \ldots + \frac{\partial}{\partial q_k}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)\dot{q}_k$$

$$+ \cdots + \frac{\partial}{\partial q_f}\left(\frac{\partial \mathbf{r}_i}{\partial q_f}\right)\dot{q}_f + \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)$$

$$= \frac{\partial^2 \mathbf{r}_i}{\partial q_1 \partial q_k}\dot{q}_1 + \frac{\partial^2 \mathbf{r}_i}{\partial q_2 \partial q_k}\dot{q}_2 + \ldots + \frac{\partial^2 r_i}{\partial q_k \partial q_k}\dot{q}_k + \ldots + \frac{\partial^2 \mathbf{r}_i}{\partial t \partial q_k} \quad \ldots(12)$$

Comparing Eqs. (11) and (12),

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k} \quad \ldots(13)$$

From Eq. (10), $\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\right) = \frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\dot{\mathbf{r}}_i^2\right)$ ...(14)

Substituting (13) and (14) in Eq. (9),

$$\sum_i \sum_k m_i \left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\left(\tfrac{1}{2}\dot{\mathbf{r}}_i^2\right) - \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{r}_i}{\partial q_k}\right]\delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k$$

or $$\sum_i \sum_k \left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right) - \frac{\partial}{\partial q_k}\left(\tfrac{1}{2}m_i\dot{r}_i^2\right)\right]\delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k$$

or $$\sum_k \left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\left(\sum \frac{1}{2}m_i\dot{r}_i^2\right) - \frac{\partial}{\partial q_k}\left(\sum \frac{1}{2}m_i\dot{r}_i^2\right)\right]\delta q_k$$

$$= \sum_i \sum_k \mathbf{F}_i \cdot \frac{\partial r_i}{\partial q_k}\delta q_k \quad \ldots(15)$$

$\sum \frac{1}{2}m\dot{r}_i^2 = T =$ total kinetic energy of the system of particles ...(16)

and $$\sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \mathbf{Q}_k, \quad \ldots(17)$$

Here $\mathbf{Q}_k$'s are components of *generalised force*.

Eq. (15) becomes, $\sum_k \left\{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}\right\}\delta q_k = \sum \mathbf{Q}_k\,\delta q_k$ ...(18)

$$\therefore \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k. \quad \ldots(19)$$

This is the general form of *Lagrange's equation*. There are f such equations corresponding to f generalised co-ordinates.

When the system is wholly conservative,

$$\mathbf{F}_i = -\nabla V_i = -\frac{\partial V_i}{\partial \mathbf{r}_i}, \quad \ldots(20)$$

$$\mathbf{Q}_k = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = -\sum \frac{\partial V_i}{\partial \mathbf{r}_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = -\sum_i \frac{\partial V_i}{\partial q_k} = -\frac{\partial}{\partial q_k}\left(\sum V_i\right) = -\frac{\partial V}{\partial q_k}$$

Here $V = \sum_i V_i =$ total potential energy of the system.

Putting this value of Q_k in Eq. (19), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=-\frac{\partial V}{\partial q_k}$$

or
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}+\frac{\partial V}{\partial q_k}=0$$

or
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial}{\partial q_k}(T-V)=0. \quad \text{... (21)}$$

The potential energy V is the function of position co- ordinates q_k and not of the *generalised velocities* $\dot{q}_k$. Therefore, Eq. (21) may be written as

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}(T-V)-\frac{\partial}{\partial q_k}(T-V)=0. \quad \text{... (22)}$$

But $L=T-V$, where L is known as *Lagrangian function.*

Eq. (22) becomes,
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=0. \quad \text{... (23)}$$

This is Lagrange's equation for a *conservative system.*

6.9. Lagrange's equations for systems containing dissipative forces.

Consider a system of particles containing dissipative or frictional forces. The frictional force is proportional to the velocity of the particle, *i.e.*,

$$\mathbf{F}_i^{(d)}=-\lambda_i\dot{\mathbf{r}}_i, \quad \text{...(1)}$$

Here $\mathbf{F}_i^{(d)}$ denotes the dissipative force, $\dot{\mathbf{r}}_i$ is the velocity of i^{th} particle and λ_i is the corresponding constant of proportionality.

Forces of this type are derivable from *Rayleigh's dissipation function* R defined by

$$R=\frac{1}{2}\sum_i \lambda_i\dot{\mathbf{r}}_i^2 \quad \text{...(2)}$$

Here $i=1, 2,\ldots n$ covers all the particles of the system.

$$\frac{\partial R}{\partial \dot{\mathbf{r}}_i}=\lambda_i\dot{\mathbf{r}}_i=-\mathbf{F}_i^{(d)} \quad \text{from Eq. (1)}$$

or
$$\mathbf{F}_i^{(d)}=-\frac{\partial R}{\partial \dot{\mathbf{r}}_i} \quad \text{...(3)}$$

The Lagrange's equation in terms of $\mathbf{r}$ is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\right)-\frac{\partial L}{\partial \mathbf{r}}=\mathbf{F}_i^{(d)} \quad \text{...(4)}$$

Here term $\frac{\partial R}{\partial \dot{\mathbf{r}}_i}$ represents the dissipative force.

Lagrange's equation in generalised coordinates q_k is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=\mathbf{Q}_k^{(d)}$$

where $\mathbf{Q}_k^d$ is component of generalised force.

In order to find the Lagrange's equation in generalised co- ordinates, we have to find the components of generalised force resulting from the dissipative force.

If $Q_k^{(d)}$ is the component of generalised force along q_k, then

$$Q_k^{(d)} = \sum_i \mathbf{F}_i^{(d)} \frac{\partial \mathbf{r}_i}{\partial q_k}$$

$$= -\sum \lambda_i \dot{\mathbf{r}}_i \frac{\partial \mathbf{r}_i}{\partial q_k} \qquad \text{[from Eq. (1)]}$$

$$= -\sum \lambda_i \dot{\mathbf{r}}_i \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \qquad \left[\because \frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}\right]$$

or

$$Q_k^{(d)} = -\sum_i \lambda_i \frac{\partial}{\partial \dot{q}_k} \left(\frac{1}{2} \dot{r}_i^2\right)$$

$$= \frac{\partial}{\partial \dot{q}_k} \sum_i \left(-\frac{1}{2} \lambda_i \dot{\mathbf{r}}_i^2\right)$$

$\therefore$

$$Q_k^{(d)} = -\frac{\partial R}{\partial \dot{q}_k} \qquad \text{...(5)}$$

Therefore the Lagrange's equation for a system containing dissipative force is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = 0. \qquad \text{...(6)}$$

Hence the term $\frac{\partial R}{\partial \dot{q}_k}$ takes into account the dissipative forces.

Thus, if dissipative forces are acting on the system, we must specify two scalar functions — the Lagrangian L and Rayleigh's dissipation function R- to derive the equations of motion.

6.10. Applications of Lagrange's Equations

In order to use Lagrange's equations for the solution of a physical problem, one must use the following steps :

(*i*) Choose an appropriate coordinate system.

(*ii*) Write down the expressions for potential and kinetic energies.

(*iii*) Write down the equations of constraint, if any.

(*iv*) Choose the generalized coordinates.

(*v*) Set up the Lagrangian. $L = T - V$.

(*vi*) Solve Lagrange's equations for each generalized coordinate using, if necessary, the equations of constraint.

(a) The Atwood's machine :

Let two small heavy particles of masses M_1 and M_2 be connected by a light inextensible rope of length l passing over a frictionless light pulley (Fig. 6.2).

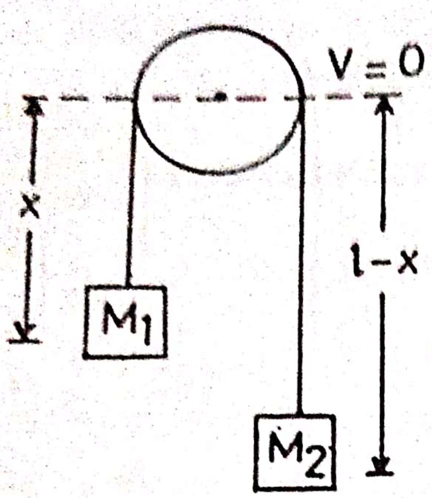

Fig. 6.2

It is found that the heavier particle descends while the lighter ascends, the system moving with a constant acceleration $\ddot{x}$.

The Atwood's machine is a conservative system with a holonomic constraint. There is only one independent coordinate x. The position of the other particle is determined by the constraint that the length of the rope between them is l.

The P.E. of the system

$$= V = -M_1 g x - M_2 g\,(l - x)$$

The K.E. of the system

$$= T = \frac{1}{2}(M_1 + M_2)\dot{x}^2$$

Hence, the Lagrangian function is given by

$$L = T - V = \frac{1}{2}(M_1 + M_2)\,\dot{x}^2 + M_1 g x + M_2 g(l - x)$$

The Lagrange's equation for a conservative system is

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = 0.$$

Since the system has only one degree of freedom, there is only one equation of motion, involving the derivatives.

∴ The equation of motion of the system is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

From the expression for L we get, $\frac{\partial L}{\partial x} = (M_1 - M_2)g$;

$$\frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\dot{x}\ ;\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}[(M_1 + M_2)\dot{x}\,] = (M_1 + M_2)\ddot{x}.$$

∴ We have, $(M_1 + M_2)\ddot{x} - (M_1 - M_2)g = 0.$

or

$$\ddot{x} = \frac{M_1 - M_2}{M_1 + M_2}g.$$

(b) A bead sliding on a Uniformly Rotating wire in a Force Free Space

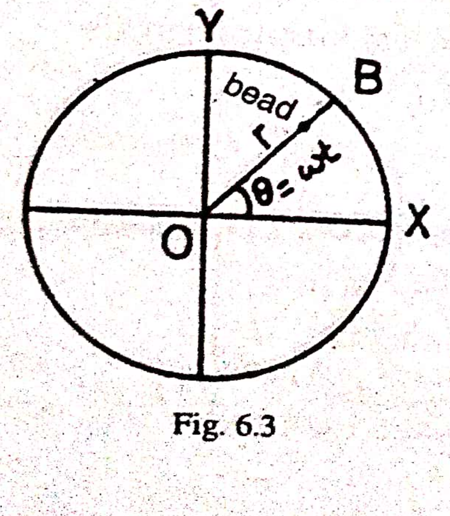

Fig. 6.3

Let OB be a straight frictionless wire fixed at a point O (Fig. 6.3). Suppose the wire rotates about a perpendicular axis through O with constant angular velocity ω. Let r be the distance of the bead from point O of the wire at time t. In this example, constraint is time dependent given by the relation $\dot{\theta} = \omega$ where ω is the angular velocity of rotation. Then the rectangular coordinates of the bead are given by

$$x = r\cos\theta = r\cos\omega t$$
$$y = r\sin\theta = r\sin\omega t$$